Relation Between Einstein And Quantum Field Equations

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Abstract

We show that there exists a choice of scalar field modes, such that the evolution of the quantum field in the zero-mass and large-mass limits is consistent with the Einstein equations for the background geometry. This choice of modes is also consistent with zero production of these scalar particles and thus corresponds to a preferred vacuum state preserved by the evolution. In the zero-mass limit, we find that the quantum field equation implies the Einstein equation that determines the scale factor for a radiation-dominated universe; in the large-mass case, it implies the corresponding Einstein equation for a matter-dominated universe. Conversely, if the classical radiation-dominated or matter-dominated Einstein equations hold, there is no production of scalar particles in the zero and large mass limits, respectively. The suppression of particle production in the large mass limit is over and above the expected suppression at large mass. Our results hold for a certain class of conformally ultrastatic background geometries and therefore generalize previous results by one of us for spatially flat Robertson-Walker background geometries. In these geometries, we find that the temporal part of the graviton equations reduces to the temporal equation for a massless minimally coupled scalar field, and therefore the results for massless particle production hold also for gravitons. Within the class of modes we study, we also find that the requirement of zero particle production of massless scalar particles or gravitons is not consistent with a non-zero cosmological constant. Possible implications are discussed.

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1 Introduction

It is a well-known prediction of quantum field theory in a curved background spacetime that the gravitational field generically creates particles. This was first shown in [1] as an effect for quantum fields propagating in a spatially flat Robertson-Walker universe. It was also shown there that the requirement of zero particle production by a spatially flat expanding universe implies the Einstein equation, with zero cosmological constant, for the scale factor, in two limiting cases. In the first case, the requirement of zero production of massless minimally coupled particles, along with the assumption that the basic mode functions have their simplest possible form, leads to the Einstein equation for the scale factor of a radiation-dominated universe. In the second case, the requirement of zero production of highly massive minimally coupled particles, with the same mode functions, leads to the Einstein equation for the scale factor of a matter-dominated universe. Conversely, this means that there is precisely no creation of massless particles in a radiation-dominated spatially flat universe, and precisely no creation of highly massive particles in a matter-dominated (or dust-filled) one.

These results show a consistency between the quantized matter field equations and the Einstein equations, though each is formally independent of the other, and suggest that the full quantized equations governing matter may, in certain circumstances, imply the macroscopic Einstein equations [1, 2]. Based on these results, it was conjectured that a gravitational Lenz's law may hold true [1, 3]. In other words, the backreaction of particle creation and vacuum energy will modify the geometry in such a way as to reduce the creation rate, eventually bringing it down to zero, when a self-consistent equilibrium-like state of matter and spacetime is reached in which no further particle creation occurs. In the low- and high-mass limits, it appears from the results of Ref. [1] that such self-consistent states exist in the context of a quantum field theory of particles. For example, if the universe is saturated with a large number of massless particles with equation of state $p = (1/3)\rho$, there is no further creation of massless particles; and similarly for the highly massive particles with equation of state $p = 0$ [1, 4]. It is of interest to examine whether the results of Ref. [1] may be generalized in the context of a quantum field theory of particles to a wider class of spacetimes. This paper carries out such a generalization.

We would like to mention other related work on similar problems. In [3] it was shown that the requirement of zero production of massless particles implies the Einstein equations with zero cosmological constant. We will derive a more general version of that result in this paper. The works of Ref. [5] suggest other related mechanisms within field theory which effectively damp the cosmological constant to zero, after which the universe enters a Robertson-Walker phase. Also, the requirement of proportionality between entropy transfer across a local Rindler horizon and its area has been shown to imply the Einstein equations [6].

Let us first summarize briefly the conclusions reached in Ref. [1]. There, the class of geometries considered are the spatially flat Robertson-Walker universes, with metric

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2), \tag{1.1}$$

with $a(t)$ being an arbitrary positive function, the scale factor of the universe.

Consider a minimally coupled scalar field propagating in this background geometry, with the usual Lagrangian density

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + m^2\phi^2\right), \tag{1.2}$$

giving rise to the equations of motion

$$(g^{\mu\nu}\nabla_\mu\nabla_\nu - m^2)\phi = 0. \tag{1.3}$$

We now separate variables and decompose the field in a box of volume V, to obtain

$$\phi(\mathbf{x},t) = (2Va^3(t))^{-\frac{1}{2}}\sum_{\mathbf{k}}[A_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}u_k(t)^* + \text{H.c.}]. \tag{1.4}$$

The choice of basis mode functions $u_k(t)$ fixes the annihilation operators $A_{\mathbf{k}}$, and hence the vacuum state under consideration. They satisfy the equation

$$\ddot{u}_k + \left[a^{-2}k^2 - \frac{3}{4}(a^{-1}\dot{a})^2 - \frac{3}{2}a^{-1}\ddot{a} + m^2\right]u_k = 0. \tag{1.5}$$

It is useful to define the effective mode frequency $\omega_k(t) = (a^{-2}k^2 + m^2)^{\frac{1}{2}}$, in terms of which the elementary (negative frequency) solutions to Eq.(1.5) reduce to $\exp(i\omega_k t)$ when $a(t)$ is constant.

For a general function $a(t)$, no particle creation will occur for asymptotically static expansions if the exact mode functions are of the form

$$u_k(t) = W_k(t)^{-\frac{1}{2}}\exp\left(i\int^t dt' W_k(t')\right), \tag{1.6}$$

where the functions W_k are such that they reduce to ω_k whenever $a(t)$ is constant. We are not requiring that the physical $a(t)$ be asymptotically static, but are imagining joining two asymptotically static regions in order to analyze the physical particle content. This is analogous to adiabatically turning on and off the interaction in elementary particle scattering theory.

The mode equation (1.5) with (1.6) implies that W_k satisfies the following equation:

$$W_k^2 - W_k^{\frac{1}{2}}\frac{d^2}{dt^2}W_k^{-\frac{1}{2}} - a^{-2}k^2 + \frac{3}{4}(a^{-1}\dot{a})^2 + \frac{3}{2}(a^{-1}\ddot{a}) - m^2 = 0. \tag{1.7}$$

The is the equation of evolution for W_k. If we now constrain the form of W_k such that it is the *simplest* form which reduces to ω_k when $a(t)$ is constant, i.e. $W_k(t) = \omega_k(t)$, then the above equation becomes an equation for $\omega_k(t)$, and hence a condition on the scale factor $a(t)$. Let us see how this comes about.

Substituting $W_k(t) = \omega_k(t)$ in (1.7), we get the equation

$$C_1(k,t)(a^{-1}\dot{a})^2 + C_2(k,t)a^{-1}\ddot{a} = 0, \tag{1.8}$$

where

$$\begin{aligned} C_1(k,t) &= \frac{k^4 + 3m^2k^2a(t)^2 + (3/4)m^4a(t)^4}{(k^4 + m^2a(t)^2)^2}, \\ C_2(k,t) &= \frac{k^2 + (3/2)m^2a(t)^2}{k^2 + m^2a(t)^2}. \end{aligned} \tag{1.9}$$

In order that the condition (1.8) be true for all values of k and t, C_1 and C_2 must be independent of k. This happens only for two choices of the mass m, namely, $m = 0$ and in the limit $m \to \infty$.

In the massless case ($m = 0$), Equation (1.8) reduces to the condition

$$(a^{-1}\dot{a})^2 + a^{-1}\ddot{a} = 0. \tag{1.10}$$

This is recognized as one of the Einstein equations for a radiation dominated cosmology, with solution $a(t) \sim t^{1/2}$.

On the other hand, in the highly massive case ($m^2 \gg k^2$), Equation (1.8) reduces to the condition

$$\frac{3}{4}(a^{-1}\dot{a})^2 + \frac{3}{2}a^{-1}\ddot{a} = 0, \tag{1.11}$$

which is one of the Einstein equations for a matter dominated (or dust-filled) universe, with solution $a(t) \sim t^{2/3}$.

The Einstein equation obtained from the zero-particle-creation condition, in each case, is the one that follows by elimination of the energy density and pressure from the full set of Einstein equations using the equation of state. We will call this the purely geometric Einstein equation. Note that the equations of state also follow from the quantum field equation in the appropriate limits [7].

To summarize, we see that there exists a certain choice of the function W_k consistent with zero particle creation, which reduces to the correct form in flat space, and for which the minimally coupled scalar field equations imply the purely geometric Einstein equation in two limits on the mass. In the massless case, a radiation-filled universe is implied, and for the highly massive case, a dust-filled one. This surprising result lends support to the gravitational Lenz's law conjecture discussed earlier. The mode functions corresponding to this choice of W_k therefore define quantum states whose particle content is preserved by the evolution, in the zero- and high-mass limits, when the radiation- and matter-dominated purely geometric Einstein equation holds, respectively. We will call such states gravitationally-preferred states. It seems reasonable to assume that the mode functions in these gravitationally-preferred states define physical particles. Here, the Einstein equations replace the timelike symmetry used in Minkowski space to define physical particles. We will discuss this further in a later paper [8].

In this paper we will show that such states exist more generally, for scalar fields with arbitrary coupling to curvature, and for gravitons, in closed and open Robertson-Walker cosmologies, and in a more general class of conformally ultrastatic spacetimes. In each case we shall show that there exists a set of modes of the form in Eq.(1.6)(corresponding to zero particle creation), which, when identified as the exact modes of the scalar field equation, constrain the geometry to satisfy the purely geometric Einstein field equation.

The organization of the paper is as follows. In Section 2, we derive the flat Robertson-Walker result in a different way, explicitly making the connection between the choice of vacuum and the condition on the geometry, and generalizing this result to the case of arbitrary coupling to scalar curvature. In Section 3, we see how the method of constructing modes in Section 2 may be used to treat the closed and open Robertson-Walker (RW) universes. In Section 4, we show that this method can be generalized to treat a certain class

of conformally ultrastatic spacetimes, of which the RW family is a special case. In Section 5, we show that the Einstein equations with non-zero cosmological constant are inconsistent with zero creation of massless particles for the general class of geometries considered in Section 4. In the case of highly massive particles, this conclusion can be reached only if the cosmological constant itself does not appear in the fundamental solutions to the quantum field equations. Finally, in Section 6, we show that the graviton equations can be separated in this class of geometries, and that their temporal part reduces to the temporal equation for a massless minimally coupled scalar field, thus enabling one to make similar conclusions for gravitons. This reduction to a minimally coupled scalar field was carried out by Lifshitz [9] for gravitons in RW spacetimes, and is here generalized to background metrics of the form in Eq. (4.10). Certain properties of the Einstein equations and a proof of consistency of the transverse-traceless-harmonic gauge for gravitons are dealt with in the appendices.

2 Spatially Flat RW with Curvature Coupling

In the Introduction, we obtained the Einstein equations for a flat RW background by making the simplest choice for W_k which was consistent with zero particle production for asymptotically static expansion of the universe. Here we will rederive these results, generalized to include arbitrary curvature coupling of the scalar field, but by asking the question: for what choice of W_k do the scalar field equations imply the purely geometric Einstein equation? To answer this, we will introduce a parametrized family of possible forms of W_k consistent with zero particle production for asymptotically static expansions and constrain the value of the parameter such that the Einstein equations hold.

We therefore begin by considering a free quantized scalar field $\phi(x)$ propagating in a spatially flat RW spacetime with metric (1.1), with an arbitrary scale factor $a(t)$. The field operator now satisfies the Klein-Gordon equation with curvature coupling:

$$(g^{\mu\nu}\nabla_\mu\nabla_\nu - m^2 - \xi R)\phi = 0, \tag{2.1}$$

The field may be quantized in the usual manner, leading to the mode expansion

$$\phi(\mathbf{x}, t) = (2\pi)^{-3/2} \int d^3k \, (A_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{x}} v_k(t) + \text{H.c.}), \tag{2.2}$$

where the continuum limit has been taken in the mode expansion, and the mode functions $v_k(t)$ differ from the functions $u_k(t)$ in the Introduction by a factor of $a^{-3/2}$. They satisfy the equation

$$\ddot{v}_k + 3\frac{\dot{a}}{a}\dot{v}_k + (k^2 a^{-2} + m^2 + \xi R)v_k = 0. \tag{2.3}$$

Since RW spacetimes are conformally flat, it is convenient to work with the conformal time η, defined by

$$\eta = \int^t dt \, a^{-1}(t), \tag{2.4}$$

and also define the conformal factor $C(\eta) = a^2(\eta)$. With these redefinitions, the metric is given by $ds^2 = C(\eta)(-d\eta^2 + \sum_i dx^i dx^i)$. Furthermore, a redefinition of the mode functions, $\chi_k = C^{1/2} v_k$, eliminates the first derivative term in Equation (2.3), and yields

$$\chi_k'' + \Omega_k^2(\eta)\chi_k = 0 \tag{2.5}$$

where a prime denotes derivative with respect to the conformal time η. It is useful to define the effective mode frequency Ω_k by

$$\Omega_k = \left[k^2 + C\left(m^2 + \left(\xi - \frac{1}{6}\right) R\right)\right]^{1/2}. \tag{2.6}$$

Conditions on quantum field behavior are now imposed by demanding a particular form for the solutions to Equation (2.5). We will require that these mode functions are given by

$$\chi_k = (2W_k)^{-1/2}(\eta) \exp\left(i \int^\eta d\eta\, W_k(\eta)\right) \tag{2.7}$$

where

$$W_k = (\Omega_k^2 + \alpha CR)^{1/2}. \tag{2.8}$$

Note that this definition of W_k differs from the one in the Introduction by a factor of a^{-1}. Here, α is a dimensionless constant parametrizing the family of mode solutions, or equivalently, the family of vacuum states, to be left arbitrary for now. Also, when $\xi = 0$ and $\alpha = 1/6$, $W_k = (k^2 + m^2a^2)^{1/2} = a\omega_k$, which was the simple form discussed in the Introduction. We will later find that α is constrained to take this value for all ξ by the requirement that the modes satisfy Equation (2.5), and by consistency with the classical Einstein equations. The above family of possible solutions to Equation (2.5) satisfies the zero particle creation condition, because, when $a(t)$ is constant, χ_k becomes a pure negative frequency mode.

We will now demand that χ_k satisfy exactly the field equation (2.5). We first obtain

$$\chi_k'' + \Omega_k^2\chi_k = \left[\frac{3}{4}\frac{W_k'^2}{W_k^2} - \frac{1}{2}\frac{W_k''}{W_k} - W_k^2 + k^2 \; + C\left(m^2 + \left(\xi - \frac{1}{6}\right) R\right)\right]\chi_k. \tag{2.9}$$

The right hand side of the above equation must vanish identically for χ_k to be a mode solution. Substituting for W_k from Eq. (2.8), this implies

$$\begin{aligned} &\frac{5}{16}W_k^{-4}\left[C'(m^2 + \zeta R) + \zeta CR'\right]^2 \\ &\quad -\frac{1}{4}W_k^{-2}\left[C''(m^2 + \zeta R) + 2\zeta C'R' + \zeta CR''\right] - \alpha CR = 0, \end{aligned} \tag{2.10}$$

where

$$\zeta = \xi + \alpha - \frac{1}{6}. \tag{2.11}$$

We will use Eq. (2.10) to obtain a condition on the geometry in the two cases when the field is massless, and when its mass is very large.

Zero mass case

For massless fields, Eq. (2.10) leads to the condition

$$\frac{5}{16}\zeta^2\left(\frac{d}{d\eta}(CR)\right)^2 - \frac{1}{4}\zeta W_k^2\frac{d^2}{d\eta^2}(CR) - \alpha CRW_k^4 = 0 \tag{2.12}$$

We wish to fix the constant α such that the above equation holds for all k. This means that the coefficient of every power of k must vanish separately. A little algebra then shows that the only possible solutions are: (i) $\alpha = 0$ and $(\xi - 1/6)CR =$ constant, or (ii) $\alpha \neq 0$ and $R = 0$. Therefore if we choose some value of $\alpha \neq 0$, we must have the condition $R = 0$ on the geometry, which is one of the Einstein equations for a radiation-dominated universe. On the other hand, if we choose $\alpha = 0$, there is no condition on the geometry at conformal coupling $\xi = 1/6$. This happens because the modes in this case are the exact negative-frequency modes at conformal coupling, independent of the geometry. However, $\alpha = 0$ is certainly consistent with the Einstein equation $R = 0$ for any value of ξ, although not a sufficient condition for it.

Thus we may conclude from the preceding analysis, that, *for any value of ξ, the choice $\alpha \neq 0$ implies the classical Einstein equation determining the scale factor for a radiation-dominated universe. All choices of α are consistent with this equation.* If we assume that α is independent of the mass, then the high-mass limit considered below will turn out to exclude $\alpha = 0$.

Large mass case

In the limit of mass approaching infinity, we may expand Eq.(2.10) in inverse powers of the mass. The leading term in such an expansion is a term of order m^2 in Eq.(2.10) whose coefficient vanishes after substituting for W_k in that equation. Therefore the leading term in Eq.(2.10) is of order 1, and the corrections are of order m^{-2}. Keeping order 1 and order m^{-2} terms in Eq.(2.10) yields the condition

$$\begin{aligned} \frac{5}{16}C^{-2}C'^2 - \frac{1}{4}C^{-1}C'' - \alpha CR + \frac{1}{4m^2C}\Bigg[\frac{1}{2}\left(\xi + \alpha - \frac{1}{6}\right)C^{-1}C'R' \\ - \left(\xi + \alpha - \frac{1}{6}\right)R'' + k^2\left(C^{-2}C'' - \frac{5}{2}C^{-3}C'^2\right)\Bigg] = 0. \end{aligned} \tag{2.13}$$

If we ignore next to leading corrections in the above equation, we find that the dominant contribution is a pure condition on the geometry, independent of mode number. Substituting for R (Eq.(A.13) with $K = 0$), this condition yields

$$\left(\frac{5}{16} + \frac{3}{2}\alpha\right)C^{-2}C'^2 - \left(\frac{1}{4} + 3\alpha\right)C^{-1}C'' = 0, \tag{2.14}$$

which is the correct Einstein equation for a matter-dominated flat RW universe, if $\alpha = 1/6$. This corresponds to a W_k of the form

$$W_k^2 = k^2 + C(m^2 + \xi R). \tag{2.15}$$

The addition of a ξR term is related to a shift in the physical mass [8]. In the minimally coupled ($\xi = 0$) case, this is the simplest choice of W_k consistent with zero particle creation. Furthermore, at minimal coupling, ζ also vanishes, and the conditions $m^2 \gg \zeta R$, etc., necessary for the validity of the high-mass expansion of Eq. (2.10), are automatically satisfied. Therefore there is no condition on the magnitude of the curvature scalar for the high mass results to hold at minimal coupling.

Note that we chose the coefficient of the m^2 term in the definition of W_k (Eq.(2.8)) to be the same as the coefficient of the m^2 term in Ω_k. This ensures that the leading term in

Eq.(2.10) above is of order 1 rather than order m^2. The fact that the coefficient of the m^2 term in (2.10) vanishes means that, as one would expect, there is no particle creation in the strict $m \to \infty$ limit, independent of the geometry. The constraint (2.13) on the geometry is therefore a next to leading order effect in the mass.

We thus conclude that *for any value of ξ, the choice $\alpha = 1/6$ in W_k, which is consistent with zero creation of highly massive particles, implies the Einstein equation determining the scale factor for a matter-dominated universe.*

Furthermore, if α is independent of mass, then this non-zero value of α also implies that the radiation-dominated Einstein equation $R = 0$ must hold in the massless case (for any ξ).

We now show that the choice of W_k in the limit of large mass is unique: that is, we do not need to assume the form in Eq.(2.8) for W_k to begin with. In the large mass limit one may assume the general form $W_k = m(f(\eta) + g(\eta)m^{-2} +$ terms of order $m^{-4})$, where $f(\eta)$ and $g(\eta)$ are unknown functions. We now substitute this expression for W_k into the right-hand-side (RHS) of Eq.(2.9), and set it to zero. The resulting differential equation, to order 1 in powers of m^{-2}, is

$$m^2(C - f^2) + \frac{3}{4}\frac{f'^2}{f^2} - \frac{1}{2}\frac{f''}{f} - 2f\,g + k^2 + \left(\xi - \frac{1}{6}\right) CR = 0. \tag{2.16}$$

Setting the coefficient of m^2 to zero yields $f(\eta) = C^{1/2}(\eta)$. Setting the coefficient of m^0 to zero and using the purely geometric Einstein equation (A.21) with $K = 0$, yields $2g(\eta) = C^{-1/2}(k^2 + \xi CR)$. Therefore, at large mass, W_k takes the form

$$W_k = mC^{\frac{1}{2}}\left[1 + \frac{1}{2}m^{-2}C^{-1}(k^2 + \xi CR)\right] + \mathcal{O}(m^{-3}), \tag{2.17}$$

which agrees with a high-mass expansion of W_k as given by Eq.(2.15). In particular, at minimal coupling, it agrees with the form for the mode functions given in the Introduction.

3 Open and Closed RW Spacetimes

For RW spacetimes with constant spatial curvature, we shall need a slight modification of the fundamental modes χ_k in order to generate the Einstein equations in the two limiting cases of high and zero mass.

The spacetime metric is now given by

$$ds^2 = C(\eta)(-d\eta^2 + p_{ij}dx^i dx^j), \tag{3.1}$$

where

$$p_{ij}dx^i dx^j = (1 - Kr^2)^{-1}dr^2 + r^2 d\Omega^2, \tag{3.2}$$

with $K = \pm 1$ and 0 for the closed, open and flat spatial geometries. We use Latin indices to denote spatial components. Defining $p \equiv \det(p_{ij})$, the scalar field equation with curvature term becomes

$$C^{-1}\partial_\eta(C\partial_\eta)\phi - p^{-\frac{1}{2}}\sum_{i,j}\partial_i(p^{\frac{1}{2}}p^{ij}\partial_j)\phi + C(m^2 + \xi R)\phi = 0. \tag{3.3}$$

This may be solved by separation of variables. Therefore we expand the scalar field in terms of the modes

$$\phi(\mathbf{x}, t) = (2\pi)^{-3/2} \int d\mu(k)(A_{\mathbf{k}}\, Y_k(\mathbf{x}) v_k(\eta) + \text{H.c.}), \tag{3.4}$$

where the spatial modes Y_k are eigenfunctions of the Laplacian on the three-space [7]. The definition of the measure $\mu(k)$ for the open, closed and flat cases may also be found in the same reference.

The spatial modes satisfy the equation

$$\Delta^{(3)} Y_k \equiv p^{-\frac{1}{2}} \partial_i (p^{\frac{1}{2}} p^{ij} \partial_j) Y_k = (K - k^2) Y_k, \tag{3.5}$$

where the quantum numbers k are defined such that $k^2 - K \geq 0$ for all three values of K.

The above equation implies that the temporal modes satisfy

$$C^{-1} \partial_\eta (C \partial_\eta) v_k + (C(m^2 + \xi R) + k^2 - K) v_k = 0. \tag{3.6}$$

Introducing the conformal modes $\chi_k = C^{\frac{1}{2}} v_k$ as before, this equation reduces to the equation

$$\chi_k'' + \left[k^2 + \left(m^2 + \left(\xi - \frac{1}{6} \right) R \right) C \right] \chi_k = 0. \tag{3.7}$$

As in the previous section, we will construct solutions to the above equation which reduce to exact negative frequency modes whenever $C(\eta)$ is constant (and thus correspond to zero particle creation), and generate the Einstein equations at large mass and zero mass. It turns out that the form in Eq.(2.7) is sufficient for this purpose, with a slightly different choice for the functions W_k. We will now require

$$W_k = \left(k^2 + (m^2 + \xi R) C - \beta(m^2) K \right)^{\frac{1}{2}}, \tag{3.8}$$

where $\beta(m^2)$ is some dimensionless function of the mass, and independent of the conformal factor. We require that W_k be real, i.e. $k^2 - \beta(m^2) K \geq k^2 - K \geq 0$, i.e. $\beta(m^2) \leq 1$. The precise values of β in the two mass limits of interest will be determined later, by the requirement of consistency with the Einstein equations. For now we will only assume that it varies very slowly at large mass, so that its derivatives do not contribute to the first few orders in a large mass expansion, which we shall carry out below.

Demanding that these modes now satisfy Equation (3.7) leads to the condition

$$\frac{3}{4} W_k^{-2} W_k'^2 - \frac{1}{2} W_k^{-1} W_k'' - \frac{1}{6} C R + \beta(m^2) K = 0. \tag{3.9}$$

In the massless ($m = 0$) case, the above condition implies

$$\frac{5}{16} \xi^2 \left(\frac{d}{d\eta} (CR) \right)^2 - \frac{1}{4} \xi W_k^2 \frac{d^2}{d\eta^2} (CR) - W_k^4 \left(\frac{1}{6} C R - \beta(0) K \right) = 0. \tag{3.10}$$

The most general solution to the above equation which is independent of k is $CR = 6\beta(0)K$. For this solution to be the purely geometric Einstein equation, $R = 0$, for a radiation dominated universe, we must have $\beta(0) = 0$. This choice for $\beta(0)$ is also motivated by the

fact that it must be zero in the massless conformally coupled case for the modes χ_k to reduce to the exact solutions in that limit. Note that when $m = 0$, these results mean that the physical mass remains zero.

To analyze the high mass limit, we may expand (3.9) in powers of m^{-2}, to get

$$\frac{5}{16}C^{-2}C'^2 - \frac{1}{4}C^{-1}C'' - \frac{1}{6}CR + \beta(\infty)K + \mathcal{O}(m^{-2}) = 0. \tag{3.11}$$

Substituting for the scalar curvature R (see Eq.(A.13)), this yields

$$\frac{3}{4}C^{-2}C'^2 - C^{-1}C'' - \frac{4}{3}(1 - \beta(\infty))K = 0, \tag{3.12}$$

which is the purely geometric Einstein equation (A.21) for a matter-dominated RW universe, if $\beta(\infty) = 1/4$.

To summarize, we therefore find that the choice (3.8) for W_k leads to the correct geometric Einstein equations in the two limiting cases, provided $\beta(m^2)$ takes the values $\beta(0) = 0$, and $\beta(\infty) = 1/4$. This restricts the form of W_k in these two limits. In either limit, it is true that $k^2 - \beta K \geq 0$, therefore ensuring that W_k is real when C is constant.

4 Conformally Ultrastatic Spacetimes

We would now like to generalize the previous results by understanding them better. The purely geometric Einstein equations considered so far have generalizations to arbitrary spacetimes. The geometric equation for a radiation-dominated spacetime continues to be $R = 0$. For an arbitary matter-dominated spacetime, the corresponding equation, obtained by eliminating the pressure p from the Einstein equations using the equation of state $p = 0$, is $R_{\mu\nu}u^\mu u^\nu = (1/2)R$, where u^μ is the four-velocity of the fluid elements, and therefore a geodesic tangent vector. This equation is derived in the Appendix, Eqs.(A.10-12).

The behavior of the field equations at zero mass seems plausible because the chosen modes do not correspond to exact solutions when $\xi = 1/6$ (i.e. conformal modes) unless $R = 0$. Hence, the requirement of zero particle creation at zero mass constrains the geometry to satisfy $R = 0$. However, this sort of argument does not explain the behavior when the mass is large. In that limit, there exists a choice of the dimensionless parameters in the mode functions for all three types of RW geometries, such that the scalar field equations imply the purely geometric Einstein equation $R_{\mu\nu}u^\mu u^\nu = (1/2)R$. The appearance of $R_{\mu\nu}u^\mu u^\nu$ in the scalar field equations is rather mysterious because the scalar field equations do not seem to depend on timelike geodesics.

However, in Robertson-Walker spacetimes, it is easy to see that the geodesic tangent vector field $u \equiv d/dt$ is proportional to the conformal killing vector field $d/d\eta$ which generates translations of conformal time. Directional derivatives along the orbits of this vector field appear explicitly in the scalar field equations. We thus have a handle on how to obtain a generalization of the Robertson-Walker results. We must look for spacetimes with a timelike conformal killing vector field, and require that the integral curves of this vector field are geodesics.

We therefore consider a general spacetime with an everywhere timelike vector field b^μ, and pick the time coordinate η to be the affine parameter along integral curves of b. With this choice, we have $b^\mu = (1, 0, 0, 0)$. We now demand that b be a conformal killing vector field i.e. the spacetime is conformal to a stationary spacetime. We will call such a spacetime "conformally stationary". This property implies

$$2\nabla_{(\mu} b_{\nu)} = \lambda(x) g_{\mu\nu}, \tag{4.1}$$

where $\lambda(x)$ is some function. In the chosen coordinate system, the left hand side of the above equation reduces to $g_{\mu\nu,0}$. We thus get

$$g_{\mu\nu,0} = \lambda(x) g_{\mu\nu}, \tag{4.2}$$

and therefore, in these coordinates, the metric takes the form

$$g_{\mu\nu}(x) = f_{\mu\nu}(\mathbf{x}) \exp\left(\int^\eta d\eta \lambda(x)\right), \tag{4.3}$$

where $\mathbf{x}$ collectively denotes the spatial coordinates. This is just a restatement of the conformally stationary property.

We further require that b is tangent to a geodesic, and therefore satisfies the equation

$$b^\mu \nabla_\mu b_\nu = \tau(x) b_\nu. \tag{4.4}$$

Although a rescaling of b by a scalar quantity would lead to an affinely parametrized geodesic equation, such a rescaling would change the form of Eq. (4.1). We find it convenient to keep the form of Eq. (4.1) unchanged and allow for a non-affine parametrization in Eq. (4.4) above. Applying b^μ to Equation (4.1), and using Eq. (4.4), we obtain

$$\frac{1}{2}\partial_\nu(b^2) = (\lambda(x) - \tau(x)) b_\nu. \tag{4.5}$$

In the chosen coordinate system, $b_\nu = g_{\nu 0}$. Therefore $b^2 = g_{\mu\nu} b^\mu b^\nu = g_{00}$, and we get the equation

$$\frac{1}{2}\partial_\nu(g_{00}) = (\lambda(x) - \tau(x)) g_{\nu 0}. \tag{4.6}$$

Using Eq. (4.3), we find that the time component of Eq. (4.6) yields

$$\tau(x) = \frac{1}{2}\lambda(x), \tag{4.7}$$

and the spatial components then yield

$$\lambda(x) f_{i0}(\mathbf{x}) = \partial_i f_{00}(\mathbf{x}) + f_{00}(\mathbf{x})\, \partial_i \int^\eta d\eta \lambda(x), \tag{4.8}$$

where we have used (4.7) in deriving the above equation.

We will now consider a subclass of the metrics which satisfy equations (4.7) and (4.8) above. A restriction which will render the field equation separable but is still more general than the RW family of spacetimes, is to require that the vector field b is orthogonal to the

hypersurfaces of constant conformal time η. Then the spacetime is conformally static, and the metric components f_{i0} can be chosen to vanish. With this staticity assumption, Eq. (4.8) implies

$$\exp\left(\int^{\eta} d\eta\lambda(x)\right) = f_{00}^{-1}(\mathbf{x})C(\eta), \tag{4.9}$$

where $C(\eta)$ is an arbitrary function of the conformal time. The metric may therefore be written as

$$ds^2 = C(\eta)[-d\eta^2 + p_{ij}(\mathbf{x})dx^i dx^j], \tag{4.10}$$

where $p_{ij} = f_{00}^{-1} f_{ij}$ are now arbitrary functions of the spatial coordinates. The Robertson-Walker family of metrics are the special cases corresponding to three-spaces of constant curvature.

We have therefore shown that the most general conformally static metrics whose conformal killing vector field is tangent to a geodesic, are the class of conformally ultrastatic metrics given by (4.10).

Our next step is to consider the scalar field equation in the general spacetime given by (4.10). We first note that CR is a sum of function of time and a function of the spatial coordinates (see Eq.(A.7)). This separability, combined with the fact that the metric components p_{ij} are functions of the spatial coordinates alone, implies that the scalar field equation (3.3) is separable. The field therefore admits a mode expansion of the form (3.4), with the spatial modes Y_k satisfying the equation

$$\left(\Delta^{(3)} + \xi\overline{R}\right) Y_k = -E_k Y_k, \tag{4.11}$$

where $\overline{R}$ is the scalar curvature of the ultrastatic metric conformally related to (4.10) (see Eq. (A.8)). The modes $\chi_k(\eta) = C^{\frac{1}{2}}(\eta)v_k(\eta)$ satisfy the equation

$$\chi_k'' + \left[E_k + Cm^2 + \left(\xi - \frac{1}{6}\right)(CR - \overline{R})\right]\chi_k = 0, \tag{4.12}$$

where E_k is a separation constant. Note that the quantity $CR - \overline{R}$ in the above equation is a function of time alone.

Going through the argument which is by now familiar, we demand solutions to Equation (4.12) of the form (2.7), with

$$W_k = (E_k + Cm^2 + \alpha RC - \beta(m^2)\overline{R})^{\frac{1}{2}}, \tag{4.13}$$

where α and $\beta(m^2)$ are arbitrary quantities to be determined by consistency with the Einstein equations. This choice of W_k seems inconsistent with separability of the field equation, because W_k is clearly not a function of time alone unless either $\beta = \alpha$ or $\overline{R}$ is a constant. However, we will find that, in the limits when the adiabatic modes given by (2.7) are exact (massless and high mass limits), consistency with the Einstein equations shall force $\overline{R}$ to be a constant, thus also ensuring that W_k, and hence χ_k, is a function of time alone.

The field equation (4.12) then implies an equation similar to (3.9), namely

$$\frac{3}{4}W_k^{-2}W_k'^2 - \frac{1}{2}W_k^{-1}W_k'' - \left(\alpha + \frac{1}{6} - \xi\right)CR + \left(\beta(m^2) + \frac{1}{6} - \xi\right)\overline{R} = 0. \tag{4.14}$$

Again, in the massless case, this implies

$$\frac{5}{16}\alpha^2\left(\frac{d}{d\eta}(CR)\right)^2-\frac{1}{4}\alpha W_k^2\frac{d^2}{d\eta^2}(CR)$$
$$-\left(\left(\alpha-\xi+\frac{1}{6}\right)CR-\left(\beta(0)-\xi+\frac{1}{6}\right)\overline{R}\right)W_k^4=0. \tag{4.15}$$

The most general solution of the above equation which is independent of E_k is

$$\left(\frac{1}{6}-\xi-\beta(0)\right)\overline{R}=\left(\frac{1}{6}-\xi+\alpha\right)RC, \tag{4.16}$$

with the additional condition $RC=$ constant if $\mu\neq 0$.

Therefore, we must either choose $\beta(0)=\xi-1/6$ or have $\overline{R}=0$ for consistency with the Einstein equation $R=0$. Analogous to the flat RW case treated in Section 2, any choice of α is consistent with this Einstein equation. More precisely, Eq.(4.16) *implies* the purely geometric Einstein equation for all values of $\xi\neq\alpha+1/6$. We shall now find that the value of α required for consistency with the matter-dominated Einstein equation in the large mass limit is $\alpha=\xi$. Therefore, if α is independent of the mass, this value also implies the radiation-dominated Einstein equation $R=0$ at zero mass.

In the high mass limit, Eq.(4.14) implies

$$\frac{5}{16}C^{-2}C'^2-\frac{1}{4}C^{-1}C''+CR\left(\xi-\frac{1}{6}-\alpha\right)+\overline{R}\left(-\xi+\frac{1}{6}+\beta(\infty)\right)+\mathcal{O}(m^{-2})=0. \tag{4.17}$$

Using Equations (A.7) and (A.9) to express time derivatives of C as linear combinations of CR, $\overline{R}$ and R_{00}, we get

$$\frac{1}{4}R_{00}+\left(\xi-\alpha-\frac{1}{8}\right)CR+\left(-\xi+\beta(\infty)+\frac{1}{8}\right)\overline{R}=0. \tag{4.18}$$

We still have freedom to choose α and $\beta(\infty)$. We will choose

$$\alpha=\xi,\quad \beta(\infty)=\xi-\frac{1}{8}, \tag{4.19}$$

so that Equation(4.18) becomes the purely geometric Einstein equation

$$R_{00}-\frac{1}{2}RC=0. \tag{4.20}$$

Note that this is the only choice of parameters leading to the above Einstein equation.

With the choices $\xi-1/6$ and $\xi-1/8$ for β in the two limits, $W_k=(E_k+m^2C+\xi CR+(1/8-\xi)\overline{R})^{1/2}$ at large mass, and $W_k=(E_k+\xi CR+(1/6-\xi)\overline{R})^{1/2}$ at zero mass.

We now show that, in both these limits, the relevant Einstein equation forces $\overline{R}$ to be constant, thus ensuring consistency with separation of the field equation, as mentioned earlier.

In the massless case, $R=0$ implies

$$\frac{1}{3}\overline{R}=\frac{1}{2}C^{-2}C'^2-C^{-1}C'', \tag{4.21}$$

according to Eq.(A.7). Since the left-hand-side of the above equation is a function of spatial coordinate alone, and the right-hand-side a function of time alone, each quantity must be constant, thus $\overline{R}$ is constant.

In the high mass case, $2R_{00} = CR$ implies

$$\frac{1}{3}\overline{R} = \frac{3}{2}C^{-2}C'^2 - 2C^{-1}C'', \tag{4.22}$$

where we have used Eqs.(A.7) and (A.9). Again, both sides of the above equation must be separately constant, thus $\overline{R}$ is constant.

The constancy of $\overline{R}$ in both mass limits implies that the conformal factor $C(\eta)$, in both limits, obeys equations identical to the equations for the conformal factor in the RW universes. Therefore the function $C(\eta)$ in the general conformally ultrastatic class is identical to the corresponding function in the RW universes, and this function will depend on the sign of $\overline{R}$. Nevertheless, the constancy of $\overline{R}$ does not necessarily imply a RW spacetime.

5 Inclusion of a Cosmological Constant

We now consider the question of whether the requirement of zero particle creation can be made consistent with the Einstein equations with a non-zero cosmological constant Λ. In [3], it was shown that this cannot be done for massless scalars in Robertson-Walker spacetimes. Here, we verify that result using a more general set of allowed modes, and for the general conformally ultrastatic class of spacetimes. On the other hand, we will also show that zero creation of highly massive particles can be made consistent with the Einstein equations with dust and a non-zero cosmological constant. However, this requires introduction of Λ itself into the form of W_k appearing in the mode functions of Eq. (2.7).

To this end, consider Equation (4.12) for the mode functions in the metric (4.10). We will now demand that this equation be satisfied by modes of the form (2.7), with an even more general form of W_k than we have considered so far, with more arbitrary parameters. We will allow W_k to have the form

$$W_k = \Big(E_k + C(m^2 + \gamma\Lambda) + \mu RC - \beta(m^2)\overline{R}\Big)^{\frac{1}{2}}. \tag{5.1}$$

This choice involves introducing the dimensionful cosmological constant Λ, itself, into the fundamental solutions of the field equations, even though Λ does not appear in those equations. This is a rather unnatural generalization of the form of W_k. However, in the massless case, we shall find that even this choice does not generally permit $\Lambda = 0$.

In the massless case, this choice of W_k leads to an equation similar to (4.15), namely

$$\begin{aligned}\frac{5}{16}\left(\frac{d}{d\eta}(\mu CR + \gamma C\Lambda)\right)^2 - \frac{1}{4}W_k^2\frac{d^2}{d\eta^2}(\mu CR + \gamma C\Lambda)\\ -\left(\left(\mu - \xi + \frac{1}{6}\right)CR - \left(\beta(0) - \xi + \frac{1}{6}\right)\overline{R} + \gamma C\Lambda\right)W_k^4 = 0.\end{aligned} \tag{5.2}$$

The most general solution of the above equation which is independent of E_k is given by the pair of equations

$$\left(\mu - \xi + \frac{1}{6}\right)CR + \gamma C\Lambda \;\; = \;\; \left(\beta(0) - \xi + \frac{1}{6}\right)\overline{R} \tag{5.3}$$

$$C\left(\mu R+\gamma\Lambda\right) \;=\; \kappa(\mathbf{x}) \tag{5.4}$$

where κ is an integration constant. For Eq. (5.4) to have the solution $R=4\Lambda$, which is the relevant radiation-dominated Einstein equation with a cosmological constant, we must have $\kappa=0$ and $\gamma=-4\mu$. Then Eq. (5.3) becomes

$$4\left(\frac{1}{6}-\xi\right)C\Lambda=\left((\beta(0)-\xi+\frac{1}{6}\right)\overline{R}. \tag{5.5}$$

Since the left-hand-side of the above equation is a function of time in general, and the right hand side a function of spatial coordinates, the only possibilities are:
(i) Both $\overline{R}$ and C are constant, which implies a static universe.
(ii) $\beta(0)\overline{R}=0$, $\xi=1/6$. At conformal coupling, the field equation (4.12) for $m=0$ can be explicitly solved and implies zero particle creation for any value of R. Thus the conformally coupled case is trivially consistent with $R=4\Lambda$.
(iii) $(\beta(0)+1/6-\xi)\overline{R}=0$ and $\Lambda=0$.

Therefore, if we allow only dynamical spacetimes and non-conformally coupled fields, the only possibility consistent with zero creation of massless particles is evidently $\Lambda=0$. This is a more general restatement of the results of Ref. [3]. We shall further show in the next section that gravitational perturbation of the class of conformally ultrastatic spacetimes considered here obey equations of the same form as the equation for massless minimally coupled scalar fields. Zero creation of these gravitons must then imply $\Lambda=0$ in a dynamical universe.

We will now go on to show that a non-zero value of Λ can be made consistent with zero creation of highly massive particles for a non-zero value of the parameter γ. In the high mass limit (we now have the additional requirement that $m^2\gg\Lambda$), the field equation (4.12) with W_k given by (5.1) implies an equation similar to (4.18):

$$CR\left(\xi-\mu-\frac{1}{8}\right)-\gamma C\Lambda+\frac{1}{4}R_{00}+\overline{R}\left(\beta(\infty)-\xi+\frac{1}{8}\right)=0. \tag{5.6}$$

For the above equation to reduce to the matter-dominated Einstein equation (A.15), we must then have

$$\begin{aligned}\beta(\infty) &= \xi-\frac{1}{8},\\ \mu &= \xi,\\ \gamma &= -\frac{3}{4},\end{aligned} \tag{5.7}$$

which leads to the form

$$W_k=\left(E_k+C\left(m^2+\xi R-\frac{3}{4}\Lambda\right)-\left(\xi-\frac{1}{8}\right)\overline{R}\right)^{\frac{1}{2}} \tag{5.8}$$

at high mass.

Also, an argument similar to the case of zero cosmological constant shows that $\overline{R}$ must be constant for the matter-dominated Einstein equations with cosmological constant to hold. Thus W_k is a function of time alone when the Einstein equations hold, consistent with separation of the field equation.

6 Graviton Equations

In this section, we will show that the temporal part of the graviton equations in the conformally ultrastatic metrics (4.10) are of the same form, in Transverse-Traceless-Harmonic (TTH) gauge, as the temporal equations for massless minimally coupled scalar fields. The RW case, for which this is known to be true [9, 10], is a special case of this analysis.

We begin by expressing the Einstein equations in the form

$$\begin{aligned} R_{\mu\nu} &= \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) && (6.1)\\ &= \kappa\left(u_\mu u_\nu(\rho+p) + \frac{1}{2}g_{\mu\nu}(\rho-p)\right), && (6.2)\end{aligned}$$

where the second equality holds for a perfect fluid energy-momentum tensor, given by (A.2). We have also defined $\kappa = 8\pi G$.

Consider metric perturbations $\delta g_{\mu\nu} = h_{\mu\nu}$ such that $\delta p = \delta\rho = \delta u^\mu = 0$. Unit normalization of the four-velocity u^μ then implies $h_{\mu\nu}u^\mu u^\nu = 0$. For such perturbations, we may perturb the right-hand-side (RHS) of Eq.(6.2) to first order in $h_{\mu\nu}$ to write

$$\delta R_{\mu\nu} = \kappa(\rho+p)u^\alpha u^\beta(g_{\nu\beta}h_{\mu\alpha} + g_{\mu\alpha}h_{\nu\beta}) + \frac{1}{2}\kappa(\rho-p)h_{\mu\nu}, \tag{6.3}$$

where $g_{\mu\nu}$ appearing on the RHS is the zeroth order background metric. For the rest of this section, all covariant derivatives and curvature tensors will refer to this background metric, which will also be used for raising and lowering indices.

Furthermore, the perturbed Ricci tensor, to first order in $h_{\mu\nu}$ is given by

$$\begin{aligned} \delta R_{\mu\nu} &= \frac{1}{2}(2h_{(\mu}{}^{\alpha}{}_{;\nu)\alpha} - h_{\mu\nu} - h_{\mu\nu;\alpha}{}^{\alpha})\\ &= \frac{1}{2}(h_\mu{}^\alpha{}_{;\alpha\nu} + h_\nu{}^\alpha{}_{;\alpha\mu} + 2R_{\beta(\mu}h_{\nu)}{}^\beta - 2R^\beta{}_{\mu\alpha\nu}h_\beta{}^\alpha - h_{\mu\nu} - h_{\mu\nu;\alpha}{}^\alpha) && (6.4)\end{aligned}$$

where $h = g^{\mu\nu}h_{\mu\nu}$, and we have used the Ricci identity in writing the second equality. At this point, it is useful to introduce the harmonic gauge conditions

$$h_\mu{}^\alpha{}_{;\alpha} = 0. \tag{6.5}$$

Furthermore, we show in Appendix B that it is consistent to demand the transverse traceless gauge conditions $h_{\mu\nu}u^\nu = h = 0$ for the conformally ultrastatic class of background metrics. Combining these conditions with Eqs. (6.3-5), we therefore get

$$h_{\mu\nu;\alpha}{}^\alpha - 2R_{\beta(\mu}h_{\nu)}{}^\beta + 2R^\beta{}_{\mu\alpha\nu}h_\beta{}^\alpha = \kappa(\rho-p)h_{\mu\nu}. \tag{6.6}$$

We can now use the zeroth order perfect fluid Einstein equations to express ρ and p in terms of the background curvature. This yields $\kappa(\rho-p) = (2/3)(R + R_{\mu\nu}u^\mu u^\nu)$, and Eq. (6.6) finally takes the form

$$h_{\mu\nu;\alpha}{}^\alpha - 2R_{\beta(\mu}h_{\nu)}{}^\beta + 2R^\beta{}_{\mu\alpha\nu}h_\beta{}^\alpha - \frac{2}{3}(R + R_{\alpha\beta}u^\alpha u^\beta)h_{\mu\nu} = 0. \tag{6.7}$$

We may use Eqs. (A.4-7) to explicitly evaluate the various terms in the above equation, and note that $h_{0\mu} = 0$ because of the transverse gauge. Then we get

$$h''_{mn} + C^{-1}C'h'_{mn} + \frac{2}{3}\overline{R}h_{mn} - p^{ij}\left[\overline{\nabla}_i\overline{\nabla}_j h_{mn} - 2\overline{R}_{i(n}h_{m)j} + 2\overline{R}^l{}_{min}h_{lj}\right] = 0, \tag{6.8}$$

where all barred quantities are evaluated in the ultrastatic metric (A.8). Recall that prime refers to partial derivative with respect to η. We now separate variables in Eq.(6.8) by writing $h_{mn} = \psi(\eta)H_{mn}(\mathbf{x})$. This yields the equations

$$p^{ij}\left[\overline{\nabla}_i\overline{\nabla}_j H_{mn} - 2\overline{R}_{i(n}H_{m)j} + 2\overline{R}^l{}_{min}H_{lj}\right] + \left(E_k + \frac{2}{3}\overline{R}\right)H_{mn} = 0, \tag{6.9}$$

for the spatial part, and

$$\psi'' + C^{-1}C'\psi' + E_k\psi = 0 \tag{6.10}$$

for the temporal part. Here, E_k is a separation constant. In separated variables, the harmonic gauge condition can be expressed as

$$\overline{\nabla}_j H^j{}_i = 0, \tag{6.11}$$

and the traceless condition as

$$H_i{}^i = 0. \tag{6.12}$$

The gauge conditions therefore involve only the spatial modes. The equation (6.10) for the temporal modes may be rewritten by defining the conformal modes $X(\eta) = C^{\frac{1}{2}}(\eta)\psi(\eta)$. Using Eq.(6.10), these modes can be shown to obey the equation

$$X'' + \left[E_k - \frac{1}{6}(CR - \overline{R})\right]X = 0. \tag{6.13}$$

Comparing this with Eq.(4.12) for the scalar field modes, we have therefore found that the temporal modes $X(\eta)$ satisfy the same equation as temporal modes of a minimally coupled massless scalar field. Since the analysis of massless scalar field modes in Sections 4 and 5 relied only on the temporal equation, the conclusions of that analysis also apply for gravitons. Specifically, the condition of zero graviton creation is consistent with the radiation-dominated Einstein equation *without* a cosmological constant.

7 Summary and Conclusions

To summarize our results, we have considered spatially flat RW spacetimes, spatially curved RW spacetimes, and finally a class of conformally ultrastatic spacetimes. For each case, we have shown that there exists a family of functions with the following properties:
(i) They reduce to pure negative frequency temporal mode solutions of the scalar field and graviton equations during any period when $C(\eta)$ is constant. [In particular, this holds whenever the first and second time derivatives of C vanish.]
(ii) When $C(\eta)$ is not constant, they are exact mode solutions of the scalar field and graviton equations in the zero mass limit, and the scalar field equation in the high mass limit, if

the radiation-dominated and matter-dominated Einstein equations hold, respectively; and conversely, if these mode functions are exact then the Einstein equation that determines $C(\eta)$ must hold in each mass limit.

These properties imply that there is no mixing of positive and negative frequencies (i.e. no particle creation) between any two periods with constant $C(\eta)$. For the RW family, this means that the condition of zero particle creation is consistent with the Einstein equations in the limits of zero mass and large mass, and the mode functions we have found give rise to the gravitationally-preferred states defined in the Introduction. Recall that property (i) does not require that there be actual periods in which $C(\eta)$ is static or slowly varying. Rather, property (i) is similar to the mathematical device of adiabatically turning on and off the interaction in analyzing the scattering of elementary particles. For the more general class of conformally ultrastatic spacetimes given by Eq. (4.10) with arbitrary three-metric $h_{ij}(\mathbf{x})$, it is still plausible that when the chosen temporal modes are exact, there is no particle creation due to the spatial dependence of the metric. Firstly, it does not seem possible to incorporate an event horizon for the class of conformally ultrastatic metrics considered because the black hole metrics cannot be expressed in the conformally ultrastatic form of Eq. (4.10) (in which C is only a function of time). Furthermore, in the absence of a cosmological constant, there are no de Sitter - like solutions to the radiation- and matter-dominated equations for $C(\eta)$ (such solutions would, in any event, not lead to production of real particles, although a monopole particle detector on a geodesic would be excited by vacuum fluctuations). Also, if the 3-metric h_{ij} has a scalar curvature singularity (a possible source of particle creation) at some point, it must be singular everywhere because the Einstein equations force $\overline{R}$ to be constant everywhere in the spacetime. We cannot, therefore, consider 3-metrics h_{ij} with scalar curvature singularities. In the absence of horizons and such singularities, it then seems plausible that there is no particle creation from the spatial variation of the metric[1]. We then have gravitationally-preferred states, i.e., consistency between the Einstein equations and the condition of zero particle creation in the conformally ultrastatic case as well.

In the massless case, one finds that the condition for zero particle creation for all values of ξ is $R = 0$. Note that if we restrict ourselves to $\xi = 1/6$ to begin with, then there is no particle creation even when $R \neq 0$. However, the requirement that the gravitational field giving rise to zero particle creation vary continously as a function of ξ implies that $R = 0$ also for $\xi = 1/6$. It is worth pointing out that the conformally ultrastatic metrics are not, in general, conformally flat. Nevertheless, we find that the mode equation (4.12) at $m = 0$ and $\xi = 1/6$ (conformal coupling) can be solved exactly and implies zero particle creation for any value of R. This result is surprising because the usual proof of zero particle creation at conformal coupling and zero mass makes use of conformal flatness, which is not present in this case.

Also, it was shown in [3] that zero creation of massless scalars or gravitons in RW universes is not consistent with a non-zero cosmological constant. That conclusion was, however, based on a rather restrictive assumption about the form of the mode functions. The analysis of Section 5, on the other hand, comes to the same conclusion by allowing for more param-

[1] We have not proved that there can be no particle creation due to the spatial variations of the metric. For example, one could certainly consider three-geometries $h_{ij}(\mathbf{x})$ such that the Riemann tensor or other curvature quantities go singular somewhere without the curvature scalar being singular. Our conclusions relating the Einstein equations to zero particle creation would not be expected to hold in such situations.

eters characterizing the modes, and holds in a more general class of spacetimes. Similarly the graviton analysis of Section 6 generalizes the known results for the RW cases to the conformally ultrastatic spacetimes.

Interpretation of the results for highly massive scalar fields is not quite so straightforward. Here, one does not expect particle production in the limit of infinite mass anyway, independent of the geometry [1]. However, the background geometry will, in general, determine the rate at which particle production vanishes as we approach infinite mass. Within the class of background geometries we have considered, we have shown that the chosen form of W_k differs from the exact form by terms of order m^{-2} only if the purely geometric Einstein equation is satisfied. If this Einstein equation does not hold, then the chosen form of W_k will differ from the exact form by terms of order 1 in an expansion in inverse powers of the mass. Therefore, if the Einstein equations do hold, the particle creation rate in the high mass limit should converge to zero faster than the expected rate for an arbitrary geometry. It should be noted that the high mass limit actually holds for any value of the curvature scalar at minimal coupling, because the condition of high mass, $m^2 \gg \xi R$, is always true for minimal coupling.

Based on the Lenz law conjecture, we speculate that the results of this paper would hold in the late time limit of a dynamical process in which the condition of zero particle creation would serve as an attractor. This can be tested by setting up a semiclassical backreaction computation whose starting point is a choice of initial geometry, and a quantum state based on a set of scalar field modes in the chosen geometry. After allowing the coupled Einstein and quantum scalar field equations to evolve, it should be possible to check if the geometry evolves to one of the forms given in this paper at late times. Hence, each mode of the scalar field will evolve to some superposition of the modes constructed in this paper; and there will be no further particle production in these modes. In order for this to happen, there must be a large amount of initial particle creation by the geometry, generating strong backreaction effects, and leading to an effectively classical energy-momentum tensor at late times. Therefore, it is necessary that we allow for large curvatures. This is consistent with our treatment, for the high mass cases with $\xi = 0$, and for the zero mass cases with any value of ξ. For high mass with nonminimal coupling, although the scalar curvature is constrained to be small for our analysis to hold, it is still possible for other curvature invariants to be large, leading to a large amount of particle production. A concrete backreaction calculation, also addressing questions of time-reversal invariance, correlations, and admissible initial states will be carried out in a later paper [8].

It is necessary to understand that the gravitational Lenz's law does *not* imply that there is no particle creation for any solution of the Einstein equations, but rather implies that the backreaction, if sufficiently large, would drive the system toward an equilibrium-like state of matter and geometry. A well-studied example is that of cosmological anisotropy damping [11]. Anisotropically evolving universes (such as the Kasner solutions) are exact solutions of the Einstein equations in which particle creation can take place. However, the backreaction of the created particles in the early stages of an anisotropic universe tends to drive the anisotropy rapidly to zero, thus inhibiting further particle creation, and leading to a Robertson-Walker spacetime. An example in which graviton production in an isotropically expanding universe gives rise to a radiation-dominated universe with no further graviton production, is treated in Ref. [12]. Another case of interest is that of an isolated black hole

emitting Hawking radiation. In this case, however, it does not seem possible to test the Lenz law conjecture because any equilibrium-like configuration is expected to occur, if at all, when the hole reaches Planck size, and semiclassical gravity theory breaks down.

Finally, we emphasize that the mode functions in a gravitationally preferred state (i.e., in one of the quasi-equilibrium states) evidently give a preferred definition of physical particles even though there is no timelike Killing vector field. This interpretation has implications to be discussed in a later work [8].

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A Perfect Fluid Einstein equations

We review the classical Einstein equations with perfect fluid matter, in the two cases when the matter is pure radiation or pure dust. In the metric signature convention $(-+++)$, the Einstein equations take the form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{A.1}$$

with the perfect fluid energy momentum tensor given by

$$T_{\mu\nu} = p g_{\mu\nu} + (p+\rho) u_\mu u_\nu, \tag{A.2}$$

where u^μ is the four-velocity of the fluid elements, and p and ρ are the principal pressure and density of the fluid respectively. A pure radiation fluid satisfies the equation of state $p = \rho/3$, and a pure dust fluid satisfies $p = 0$. We will consider these two cases separately for the class of metrics

$$ds^2 = C(\eta)(-d\eta^2 + p_{ij}(\mathbf{x}) dx^i dx^j), \tag{A.3}$$

with $\mathbf{x}$ denoting the spatial coordinates.
For this class of metrics, we may evaluate the affine connection and the various curvature tensors, to get

$$\begin{aligned}
\Gamma^\alpha{}_{\mu\nu}(x) &= \overline{\Gamma}^\alpha{}_{\mu\nu}(\mathbf{x}) + \frac{1}{2} C^{-1} C' \left(2\delta^\alpha_{(\nu}\delta^0_{\mu)} + \delta^\alpha_0 \overline{g}_{\mu\nu} \right) && \text{(A.4)}\\
R^\alpha{}_{\mu\beta\nu}(x) &= \overline{R}^\alpha{}_{\mu\beta\nu}(\mathbf{x}) + \frac{1}{2}(\delta^\alpha_{[\nu}\delta^0_{\beta]}\delta^0_\mu + \overline{g}_{\mu[\nu}\delta^0_{\beta]}\delta^\alpha_0) \left(2C^{-1}C'' - 3C^{-2}C'^2 \right) \\
&\quad + \frac{1}{2}\delta^\alpha_{[\beta}\overline{g}_{\nu]\mu} C^{-2} C'^2 && \text{(A.5)}\\
R_{\mu\nu}(x) &= \overline{R}_{\mu\nu}(\mathbf{x}) - \frac{1}{2}\delta^0_\mu \delta^0_\nu (2C^{-1}C'' - 3C^{-2}C'^2) + \frac{1}{2}\overline{g}_{\mu\nu}\, C^{-1}C'' && \text{(A.6)}\\
R(x) &= C^{-1}\overline{R}(\mathbf{x}) + 3C^{-1}\left(C^{-1}C'' - \frac{1}{2}C^{-2}C'^2 \right), && \text{(A.7)}
\end{aligned}$$

where all barred quantities are evaluated in the conformally related metric

$$d\overline{s}^2 = -d\eta^2 + p_{ij}(\mathbf{x}) dx^i dx^j, \tag{A.8}$$

and therefore depend only on the spatial coordinates. Also, note that we have defined $\overline{R}$ as the 4-dimensional scalar curvature of the metric (A.8) rather than the curvature associated with the three-metric p_{ij}. Furthermore, the only non-zero components of $\overline{\Gamma}^{\alpha}{}_{\mu\nu}, \overline{R}_{\mu\nu}$ and $\overline{R}^{\alpha}{}_{\mu\beta\nu}$ are their spatial components.
In particular, the time-time component of the Ricci tensor is given by

$$R_{00} = \frac{3}{2}\left(C^{-2}C'^{2} - C^{-1}C''\right), \tag{A.9}$$

which is independent of the three-metric p_{ij}.
Consider now the Einstein equations (A.1), first for the case $\Lambda = 0$.
For a radiation-dominated spacetime, the equation of state is $p = \rho/3$, which yields $T^{\mu}_{\mu} = 0$, and the Einstein equations then imply $R = 0$.

For a matter-dominated (dust) spacetime, the equation of state is $p = 0$. Contraction of Eq.(A.1)(with $\Lambda = 0$) then yields

$$R = -8\pi G\rho. \tag{A.10}$$

On the other hand, after multiplying Eq.(A.1) by $u^{\mu}u^{\nu}$ and summing over repeated indices, we get

$$R_{\mu\nu}u^{\mu}u^{\nu} + \frac{1}{2}R = -8\pi G\rho. \tag{A.11}$$

The two equations above together imply

$$R_{\mu\nu}u^{\mu}u^{\nu} = \frac{1}{2}R \tag{A.12}$$

for any dust-filled spacetime. For the particular class of metrics we consider, $u = C^{-1/2}d/d\eta$, and Eq.(A.12) becomes

$$R_{00} = \frac{1}{2}CR. \tag{A.13}$$

If we include the cosmological constant, the analogous equations are

$$R = 4\Lambda \tag{A.14}$$

for the radiation-dominated case, and

$$R_{\mu\nu}u^{\mu}u^{\nu} = \frac{1}{2}R - 3\Lambda \tag{A.15}$$

for the matter-dominated case.

Robertson-Walker Spacetimes

If we now assume that the spacetime is homogenous and isotropic, then it is described by the Robertson-Walker family of metrics of the form

$$ds^2 = -dt^2 + a^2(t)\left((1 - Kr^2)^{-1}dr^2 + r^2d\Omega^2\right), \tag{A.16}$$

where $K = \pm 1$ and 0 denoting the spatially closed, open and flat cases. Equivalently, in terms of the conformal time, the metric is

$$ds^2 = C(\eta)\left(-d\eta^2 + (1 - Kr^2)^{-1}dr^2 + r^2 d\Omega^2\right), \tag{A.17}$$

where $C(\eta) = a^2(\eta)$. The scalar curvature is

$$CR = 6K + 3\left(C^{-1}C'' - \frac{1}{2}C^{-2}C'^2\right), \tag{A.18}$$

and the time-time component of the Ricci tensor is given by Eq.(A.6) above.

We will now consider the Einstein equations without cosmological constant. The Einstein equation $R = 0$ for the conformal factor of a radiation-dominated universe then takes the form

$$C^{-1}C'' - \frac{1}{2}C^{-2}C'^2 = -2K, \tag{A.19}$$

or equivalently, in terms of cosmic time and the scale factor,

$$\dot{a}^2 + a\ddot{a} = -K. \tag{A.20}$$

For $K = 0$ (the flat case), one may solve the above equations to get $a(t) \sim t^{1/2}$, or $C(\eta) \sim \eta^2$.

The Einstein equation $R_{00} = (1/2)CR$ for the conformal factor of a matter-dominated universe takes the form

$$2C^{-1}C'' - \frac{3}{2}C^{-2}C'^2 = -2K, \tag{A.21}$$

or

$$\dot{a}^2 + 2a\ddot{a} = -K. \tag{A.22}$$

Again, for $K = 0$, we get $a(t) \sim t^{3/2}$, or $C(\eta) \sim \eta^4$.

B Transverse Traceless Condition for Gravitons

Here, we show that the transverse traceless (TT) condition on the gravitational perturbations is consistent with their dynamics, after fixing the harmonic gauge (6.5). This is certainly true for the RW spacetimes, as shown by Lifshitz [9] (see also [10]). We show that it is true for the general conformally ultrastatic class given by the metric (4.10).

We begin with the linearized Einstein equations in harmonic gauge, obtained by combining Eqs.(6.3-5):

$$\begin{aligned} &h_{\mu\nu;\alpha}{}^{\alpha} + h_{;\mu\nu} - R_{\beta\nu}h_{\mu}{}^{\beta} - R_{\beta\mu}h_{\nu}{}^{\beta} + 2R^{\beta}{}_{\mu\alpha\nu}h_{\beta}{}^{\alpha} = \\ &\qquad \frac{1}{2}(\rho - p)h_{\mu\nu} - (\rho + p)u^{\alpha}u^{\beta}(g_{\nu\beta}h_{\mu\alpha} + g_{\mu\alpha}h_{\nu\beta}). \end{aligned} \tag{B.1}$$

The above equation is a second order differential equation for $h_{\mu\nu}$ and its solution may be specified by specifying $h_{\mu\nu}\mid_{\eta_0}$ and its time derivative $u^{\alpha}\nabla_{\alpha}h_{\mu\nu}\mid_{\eta_0}$ on some initial spacelike hypersurface $\eta = \eta_0$. We will choose this initial data such that $h = u^{\alpha}\nabla_{\alpha}h = u^{\mu}h_{\mu\nu} = u^{\alpha}\nabla_{\alpha}(u^{\mu}h_{\mu\nu}) = 0$ initially, i.e. at $\eta = \eta_0$. We will now show that these conditions are

preserved by the equations of motion (B.1) and this will allow us to conclude that there exist dynamical perturbations of this class of spacetimes which satisfy $h = u^\mu h_{\mu\nu} = 0$ for all time, i.e. they are traceless and transverse.

We first consider the traceless condition. Taking the trace of Eq.(B.1), we get

$$h_{;\alpha}{}^{\alpha} + \frac{1}{4}(\rho - p)h = 0, \tag{B.2}$$

where we have used the condition $h_{\mu\nu}u^\mu u^\nu = 0$ demanded by the unit normalization of the four-velocity. The above equation is an equation for the trace of $h_{\mu\nu}$ which must be satisfied by every solution of Eq.(B.1). Its solution is unique once we specify the initial conditions on h and its time derivative. The chosen initial conditions $h = u^\mu \nabla_\mu h = 0$ thus imply the unique solution $h = 0$ of Eq.(B.2). Therefore there exist traceless perturbations with $h = 0$ for all time.

For traceless perturbations, the dynamical equations (B.1) reduce to the form

$$h_{\mu\nu;\alpha}{}^{\alpha} - 2R_{\beta(\nu}h_{\mu)}{}^{\beta} + 2R^{\beta}{}_{\mu\alpha\nu}h_{\beta}{}^{\alpha} = \frac{1}{2}(\rho - p)h_{\mu\nu} - (\rho + p)u^\alpha u^\beta (g_{\nu\beta}h_{\mu\alpha} + g_{\mu\alpha}h_{\nu\beta}). \tag{B.3}$$

Showing that the condition $u^\mu h_{\mu\nu} = 0$ is preserved by the above equation requires a bit more work. First, we multiply by u^μ, to get

$$u^\mu h_{\mu\nu;\alpha}{}^{\alpha} - 2u^\mu R_{\beta(\nu}h_{\mu)}{}^{\beta} + 2u^\mu R^{\beta}{}_{\mu\alpha\nu}h_{\beta}{}^{\alpha} = \frac{1}{2}(3\rho - p). \tag{B.4}$$

We wish to simplify this equation in the background metric of Eq.(4.10). In the chosen coordinate system, we then have $u^\mu = C^{-1/2}\delta_0^\mu$. Consider the first term in the above equation. We may reexpress it in the form

$$u^\mu h_{\mu\nu;\alpha}{}^{\alpha} = (u^\mu h_{\mu\nu})_{;\alpha}{}^{\alpha} - u^\mu{}_{;\alpha}{}^{\alpha}h_{\mu\nu} - 2u^\mu{}_{;\alpha}h_{\mu\nu}{}^{;\alpha}, \tag{B.5}$$

and use Eq.(A.4) for the affine connection to evaluate

$$u^\alpha{}_{;\beta} = \frac{1}{2}C'C^{-3/2}\left(\delta_\beta^\alpha - \delta_\beta^0\delta_0^\alpha\right). \tag{B.6}$$

After some straightforward but tedious simplifications, one then obtains

$$u^\mu{}_{;\alpha}{}^{\alpha} = -\frac{1}{2}\delta_0^\mu C^{-7/2}C'^2. \tag{B.7}$$

Substituting the two equations above in Eq.(B.5), and using the harmonic gauge condition, we get

$$u^\mu h_{\mu\nu;\alpha}{}^{\alpha} = (u^\mu h_{\mu\nu})_{;\alpha}{}^{\alpha} - C^{-3/2}C'u^\alpha (u^\mu h_{\mu\nu})_{;\alpha} - C^{-3}C'^2 u^\mu h_{\mu\nu}. \tag{B.8}$$

We have thus expressed the first term in Eq.(B.4) entirely in terms of $Y_\nu \equiv u^\mu h_{\mu\nu}$ and its derivatives. Similarly, we reexpress other terms, using the following relations which are easily derived using Eqs.(A.5-7):

$$u^\mu R_{\beta\mu}h_\nu{}^{\beta} = \frac{3}{2}C^{-2}(C'' - C^{-1}C'^2)Y_\nu \tag{B.9}$$

$$u^\mu R^{\beta}{}_{\mu\alpha\nu}h_{\beta}{}^{\alpha} = -\frac{1}{2}C^{-2}(C'' - C^{-1}C'^2)Y_\nu. \tag{B.10}$$

Equations (B.4) therefore reduce to differential equations for Y_ν:

$$Y_{\nu;\alpha}{}^{\alpha} - C^{-3/2}C'u^\alpha Y_{\nu;\alpha} + \frac{1}{2}C^{-2}(3C^{-1}C'^2 - 5C'')Y_\nu - R^\beta{}_\nu Y_\beta = \frac{1}{2}(3\rho - p)Y_\nu. \qquad \text{(B.11)}$$

Again the initial conditions $Y_\nu = u^\alpha Y_{\nu;\alpha} = 0$ imply the unique solution $Y_\nu = 0$ to the above hyperbolic partial differential equation. This completes the proof of consistency of the transverse traceless condition with the equations of motion for the perturbations.

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